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LETTER TO THE EDITOR

Linearizability of three families of cubic systems

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Abstract
We obtain the necessary and sufficient conditions for linearizability (isochronicity) of three families of eight-parametric cubic systems. This completes the classification of the linearizable systems studied by Romanovski and Robnik (2001 *J. Phys. A: Math. Gen.* **34** 10267–92).

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The study of isochronous oscillations goes back at least to Huygens, who investigated the motion of cycloidal pendulum. Later on, isochronous systems were studied by Euler, Bernoulli, Lagrange and others. At present, the problem of isochronicity again attracts increasing interest; see, for example, [1–4, 8, 9] and references therein.

In [8] the eight-parametric subfamilies of the cubic system

$$\begin{aligned}\dot{x} &= x(1 - a_{10}x - a_{01}y - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3)\\ \dot{y} &= -y(1 - b_{2,-1}x^2y^{-1} - b_{10}x - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2)\end{aligned}$$

such that in the first equation one of the coefficients $(a_{10}, a_{01}, a_{-12})$ is different from zero and one of the coefficients $(a_{20}, a_{11}, a_{02}, a_{-13})$ is equal to zero. The second equation is obtained from the first after the involutions

$$a_{ij} \leftrightarrow b_{ji} \qquad x \leftrightarrow y \tag{1}$$

are taken into account. There are twelve such eight-parametric systems. Ten of these cases were considered in [8], and a complete set of necessary and sufficient conditions for linearizability was obtained for nine of these. The two remaining systems are

$$\begin{aligned}\dot{x} &= x(1 - a_{01}y - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3)\\ \dot{y} &= -y(1 - b_{10}x - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy)\end{aligned} \tag{2}$$

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and

$$\begin{aligned}\dot{x} &= x(1 - a_{10}x - a_{20}x^2 - a_{02}y^2 - a_{-13}x^{-1}y^3)\\ \dot{y} &= -y(1 - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{02}y^2)\end{aligned} \tag{3}$$

which were omitted from the classification as the computational difficulties proved insuperable at the time.

In the present paper, we obtain the necessary and sufficient conditions of linearizability of a centre for both systems. Furthermore, we complete the classification of the remaining case of [8].

Recall that the origin of the real system

$$\dot{u} = -v + U(u, v) \qquad \dot{v} = u + V(u, v) \tag{4}$$

is a centre if all trajectories in its neighbourhood are closed and it is an isochronous centre if the period of oscillation is the same for all these trajectories. We assume here that $U(u, v)$ and $V(u, v)$ are analytic series without constant or linear terms.

It has been shown by Poincaré and Lyapunov that the system (4) has a formal Lyapunov first integral of the form

$$\Phi(u, v) = u^2 + v^2 + \sum_{l+j=3}^{\infty} \phi_{l,j} u^l v^j$$

if and only if the origin is a centre on the real plane u, v. Then, the integral is analytical and, according to Vorob'ev's theorem [10, 1], the centre is isochronous if and only if the system (4) is linearizable (in this case also the formal transformation is necessarily analytic).

Using the complex variables $x = u + \mathrm{i}v$ we can write the system (4) as a single equation

$$\dot{x} = \mathrm{i}(x + X(x, \bar{x})) \tag{5}$$

where $X(x, \bar{x}) = \sum_{k+l\geqslant 2} X_{kl} x^k \bar{x}^l$. It is convenient to consider $\bar{x}$ as an independent variable, $\bar{x} = y$, then from equation (5) we obtain the more general complex system

$$\dot{x} = \mathrm{i}(x + X(x, y)) \qquad \dot{y} = -\mathrm{i}(y + Y(x, y)) \tag{6}$$

where $X(x, y) = \sum_{k+l=2}^{\infty} X_{kl} x^k y^l$, $Y(x, y) = \sum_{k+l=2}^{\infty} Y_{kl} x^k y^l$, are series convergent in a neighbourhood of the origin. This system is equivalent to equation (5) when $x = \bar{y}$, $X_{ij} = \bar{Y}_{ji}$. After a change of time, $\mathrm{i}\mathrm{d}t = \mathrm{d}\tau$, we obtain the system

$$\mathrm{d}x/\mathrm{d}\tau = x + X(x, y) = \hat{X}(x, y) \qquad \mathrm{d}y/\mathrm{d}\tau = -y - Y(x, y) = \hat{Y}(x, y). \tag{7}$$

We say that system (7) is *linearizable (isochronous)* if there is an analytic change of coordinate in the neighbourhood of the origin making the system linear. We look for such transformation in the form

$$\begin{aligned} x_1 &= x + \sum_{k+j\geqslant 2} H_{kj}^{(1)} x^k y^j = \hat{H}^{(1)}(x, y)\\ y_1 &= y + \sum_{k+j\geqslant 2} H_{kj}^{(2)} x^k y^j = \hat{H}^{(2)}(x, y)\end{aligned} \tag{8}$$

so that $\dot{x}_1 = x_1$, $\dot{y}_1 = -y_1$. The functions $\hat{H}^{(1)}$, $\hat{H}^{(2)}$ must therefore satisfy the following equations:

$$\hat{H}^{(1)}(x, y) = \frac{\partial \hat{H}^{(1)}}{\partial x}\dot{x} + \frac{\partial \hat{H}^{(1)}}{\partial y}\dot{y} \qquad -\hat{H}^{(2)}(x, y) = \frac{\partial \hat{H}^{(2)}}{\partial x}\dot{x} + \frac{\partial \hat{H}^{(2)}}{\partial y}\dot{y}. \tag{9}$$

Equating the coefficients of the monomials $x^k y^j$ in these identities, we can determine uniquely the coefficients $H_{kj}^{(1)}$, $H_{jk}^{(2)}$, when $j-k \neq 1$. When $j-k = 1$ we obtain compatibility conditions $0 \cdot H^{(1)}_{k,k+1} = i_{kk}(X, Y), 0 \cdot H^{(2)}_{k+1,k} = j_{kk}(X, Y)$, where $i_{kk}(X, Y), j_{kk}(X, Y)$ are polynomials of the coefficients X_{lj}, Y_{lj} such that $l + j \leqslant 2k$; see, for example, [8] for more details. We call these polynomials the *linearizability (isochronicity) quantities*. Therefore, the system is linearizable if and only if the infinite series of the conditions

$$i_{11}(X, Y) = j_{11}(X, Y) = \cdots = i_{kk}(X, Y) = j_{kk}(X, Y) = \cdots = 0 \tag{10}$$

is satisfied. The conditions (10) are the necessary conditions of linearizability. Finding these conditions is the first step in the study of the linearizability problem. The second step is to check whether the necessary conditions obtained are also sufficient. One of the most powerful tools to carry out this second step is the Darboux linearization method.

Definition 1. *We call a change of variables*

$$x_1 = F_1(x, y) \qquad y_1 = F_2(x, y) \tag{11}$$

of the system (7), a Darboux linearization, if it transforms the system to a linear system, $\dot{x}_1 = x_1, \dot{y}_1 = -y_1$, *and is such that at least one of the functions,* F_1, F_2, *is of the form* $F = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, *where the curves* $f_i(x, y) = 0$ *determine invariant algebraic curves of the system (7), and* α_j *are complex numbers. In more detail, the polynomials* f_i *satisfy the equation*

$$\frac{\partial f_i}{\partial x}\hat{X} + \frac{\partial f_i}{\partial y}\hat{Y} = K_i f_i \tag{12}$$

for some polynomials $K_i(x, y)$, *called the co-factors of the invariant curves* $f_i(x, y) = 0$.

It is easily seen that if

$$\hat{X}(x, y)/x + \sum_{i=1}^{k} \alpha_i K_i = 1 \tag{13}$$

then the substitution $x_1 = x f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, linearizes the first equation, i.e. brings it into $\dot{x}_1 = x_1$. If

$$\hat{Y}(x, y)/y + \sum_{i=1}^{k} \alpha_i K_i = -1 \tag{14}$$

then the change $y_1 = y f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ brings the second equation of (7) to $\dot{y}_1 = -y_1$.

If, for system (7), only one of the conditions (13) and (14) is satisfied, say (14), and (7) has a Lyapunov first integral $\Psi(x, y)$ of the form

$$\Psi(x, y) = xy + \sum_{k+l=3}^{\infty} v_{kl} x^k y^l \tag{15}$$

then system (7) is linearizable by the substitution $x_1 = \Psi(x, y)/F_2(x, y), y_1 = F_2(x, y)$. Similarly, if condition (13) takes place, then the linearizing transformation is given by $x_1 = F_1(x, y), y_1 = \Psi(x, y)/F_1(x, y)$, as can be verified by a straightforward calculation [4], once it is observed that $F_2(x, y)$ $(F_1(x, y))$ must divide $\Psi(x, y)$ and so the transformations are well defined.

Given an ideal J we denote by $\mathbf{V}(J)$ the variety (the set of all common zeros of polynomials from J) of J.

Definition 2. *Let* $I = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots, i_{kk}, j_{kk}, \ldots \rangle$ *be the ideal generated by focus quantities of system (7). The variety of the ideal I,* $V_{\mathcal{L}} = \mathbf{V}(I)$, *is called the linearizability variety of system (7).*

Any system with coefficients from $V_{\mathcal{L}}$ is linearizable in a neighbourhood of the origin by a convergent substitution of the form (8).

We now tackle the remaining two cases (2) and (3). To do so, we need the following proposition.

Lemma 1. *The system*

$$\dot{x} = x(1 - x - a_{20}x^2) \qquad \dot{y} = -y(1 - y - b_{3,-1}x^3y^{-1} - b_{20}x^2) \tag{16}$$

has a first integral of the form

$$H = \sum_{m=1}^{\infty} H_m(y)x^m \tag{17}$$

where $H_1 = \frac{y}{y-1}$, *and for* $m = 6k + s \geqslant 1$

$$H_{6k+s}(y) = \begin{cases} \dfrac{H_1(y)^{6k+s}P_{5k+[\frac{s-1}{2}]}(y)}{y^{8k+s-1}} & \text{when} \quad s = 1, 2, 3 \\ \dfrac{H_1(y)^{6k+s}P_{5k+[\frac{s+1}{2}]}(y)}{y^{8k+s}} & \text{when} \quad s = 4, 5, 6 \end{cases} \tag{18}$$

(here and below we denote by P_i any polynomial of degree at most i and by $[a]$ the integer part of a).

Proof. The system (16) is similar to the system (47) from [8], namely, to

$$\dot{x} = x(1 - a_{10}x - a_{20}x^2) \qquad \dot{y} = -y(1 - b_{01}y - b_{3,-1}x^3y^{-1}) \tag{19}$$

(but system (16) contains the additional term $b_{20}yx^2$ in the second equation). It turns out that the way of construction of the integral (17) used in [8] can be transferred to the system (16).

Let us expand the equation of trajectories (16) into the power series $\frac{dx}{dy} = \sum_{i=0}^{\infty} a_i x^i$. It is easily seen that the coefficients a_i are of the form

$$a_{6k+1} = \frac{Q_{2k+1}(y)}{(1-y)^{3k+1}y^{2k+1}} \qquad a_{6k+2} = \frac{Q_{2k}(y)}{(1-y)^{3k+1}y^{2k+1}} \qquad a_{6k+3} = \frac{Q_{2k+1}(y)}{(1-y)^{3k+2}y^{2k+1}}$$

$$a_{6k+4} = \frac{Q_{2k+1}(y)}{(1-y)^{3k+2}y^{2k+2}} \qquad a_{6k+5} = \frac{Q_{2k}(y)}{(1-y)^{3k+3}y^{2k+2}} \qquad a_{6k+6} = \frac{Q_{2k+1}(y)}{(1-y)^{3k+3}y^{2k+2}}$$

where $k = 0, 1, 2, \ldots$ and $Q_m(y)$ denotes a polynomial in y of degree at most m.

The coefficients H_i of the series (17) should satisfy the differential equations

$$\begin{aligned} &H_1' + a_1H_1 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &H_k' + ka_1H_k = f_k \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{20}$$

where $f_k = -(k-1)a_2H_{k-1} - (k-2)a_3H_{k-2} - \cdots - a_kH_1$. From the first six equations we obtain

$$\begin{aligned} H_1 &= \frac{y}{y-1} & H_2 &= -\frac{H_1^2}{y} & H_3 &= \frac{H_1^3P_2(y)}{y^3} \\ H_4 &= \frac{H_1^4P_2(y)}{y^4} & H_5 &= \frac{H_1^5P_3(y)}{y^5} & H_6 &= \frac{H_1^6P_3(y)}{y^6}. \end{aligned} \tag{21}$$

We prove that the coefficients H_m have the form (18) by induction on k. According to equation (21) for $k = 0$ equation (18) holds. Let us suppose that the formula is proven for $k < m$ and consider the case $k = m$. Note that

$$H_r(y) = H_1(y)^r \int^y f_r(u) H_1(u)^{-r}\, \mathrm{d}u \tag{22}$$

where $f_r = -\sum_{i=1}^{r-1}(r-i)a_{i+1}H_{r-i}$, and

$$\int^y \frac{P_s(u)}{u^n}\, \mathrm{d}u = \frac{P_s(y)}{y^{n-1}} \tag{23}$$

when $n > s+1$. (The polynomials P_s on the right-hand and left-hand sides of (23) are different, but only the degree is important for us, so we use the same notation P_s for any polynomial of the degree s.) Using equation (22) for $k = m$ we have

$$H_{6m+s}(y) = -H_1(y)^{6m+s} \int^y \left(\sum_{i=2}^{6m+s} (6m+s+1-i) a_i(u) H_{6m+s+1-i}(u) \right) H_1(u)^{-6m-s}\, \mathrm{d}u.$$

To prove equation (18) it is sufficient to show that

$$\int^y a_{6k+l}(u) H_{6m+s+1-6k-l}(u) H_1(u)^{-6m-s}\, \mathrm{d}u = \begin{cases} \dfrac{P_{5m+[\frac{s-1}{2}]}(y)}{y^{8m+s-1}} & \text{when} \quad s = 1,2,3 \\ \dfrac{P_{5m+[\frac{s+1}{2}]}(y)}{y^{8m+s}} & \text{when} \quad s = 4,5,6. \end{cases} \tag{24}$$

for $l = 1, 2, 3, 4, 5, 6$. Let us consider the case $l = 1$. Then

$$\int^y a_{6k+1}(u) H_{6m+s-6k}(u) H_1(u)^{-6m-s}\, \mathrm{d}u = \begin{cases} \dfrac{P_{5m}(y)}{y^{8m+s-1}} & \text{when} \quad s = 1,2 \\ \dfrac{P_{5m+1}(y)}{y^{8m+s-1}} & \text{when} \quad s = 3 \\ \dfrac{P_{5m+2}(y)}{y^{8m+s}} & \text{when} \quad s = 4 \\ \dfrac{P_{5m+3}(y)}{y^{8m+s}} & \text{when} \quad s = 5,6. \end{cases} \tag{25}$$

in agreement with equation (24). Similarly, we can check that equation (24) holds for $l = 2, 3, 4, 5, 6$. □

Theorem 1.

(1) The linearizability variety of system (2) is $\mathbf{V}(\tilde{\mathcal{I}}) = \mathbf{V}(\tilde{J}_1) \cup \mathbf{V}(\tilde{J}_2)$, *where* $\tilde{J}_1 = \langle b_{10}, a_{01}b_{10} + b_{11}, b_{20}, b_{3,-1}, a_{11} - b_{11}\rangle$, $\tilde{J}_2 = \langle a_{01}b_{10} + b_{11}, a_{-13}, a_{02}, a_{11} - b_{11}, a_{01}\rangle$.

(2) The linearizability variety of system (3) consists of seven irreducible components, $\mathbf{V}(\mathcal{I}) = \cup_{i=1}^{7}\mathbf{V}(J_7)$, *where* $J_1 = \langle b_{01}, b_{20}, a_{-13}, 3a_{02} + b_{02}, a_{20}, a_{10}\rangle$, $J_2 = \langle a_{10}, b_{01}, 112b_{20}^3 + 27b_{3,-1}^2 b_{02}, 49a_{-13}b_{20}^2 - 9b_{3,-1}b_{02}^2, 21a_{-13}b_{3,-1} + 16b_{20}b_{02}, 343a_{-13}^2 b_{20} + 48b_{02}^3, 7a_{02} + 3b_{02}, 3a_{20} + 7b_{20}\rangle$, $J_3 = \langle b_{01}, b_{02}, b_{3,-1}, a_{02}, a_{20} + 3b_{20}, a_{10}\rangle$, $J_4 = \langle b_{3,-1}, a_{-13}, a_{02} + b_{02}, a_{20} + b_{20}\rangle$, $J_5 = \langle b_{20}, b_{3,-1}, a_{-13}, a_{02}\rangle$, $J_6 = \langle b_{20}, b_{3,-1}, a_{20}\rangle$, $J_7 = \langle b_{02}, a_{-13}, a_{02}\rangle$.

Proof. (1) By means of the algorithm from [8] we have computed the first twelve isochronicity quantities $i_{11}, j_{11}, \ldots, i_{66}, j_{66}$. Romanovski and Robnik [8] tried to find the irreducible decomposition of the isochronicity varieties of systems (2) and (3) using the routine *primdecGTZ* of Singular 2-0-0 [6], but they did not succeed. We use the more recent version Singular 2-0-3 and, by means of the procedure *minAssGTZ*, we found that the minimal associate primes of the ideal generated by these quantities are $\tilde{J}_1$ and $\tilde{J}_2$.

Obviously, it is sufficient to consider one of the varieties $\mathbf{V}(\tilde{J}_1)$, $\mathbf{V}(\tilde{J}_2)$, because they are mapped to each other by the involution (1). The systems from $\mathbf{V}(\tilde{J}_1)$ are of the form

$$\dot{x} = x - a_{01}xy - a_{02}xy^2 - a_{-13}y^3 \qquad \dot{y} = -y. \tag{26}$$

According to theorem 6 of [7], system (26) is time reversible for any parameter values, therefore it admits a first integral $\Psi(x, y)$ of the form (15) and because the second equation of the system is linear the system is linearizable by means of the transformation

$$x_1 = \Psi(x, y)/y \qquad y_1 = y.$$

(2) For system (3) $i_{11} = j_{11} = 0$, so we have used the ideal $\mathcal{I}_7 = \langle i_{22}, j_{22}, \ldots, i_{77}, j_{77}\rangle$. Again, making use of *minAssGTZ* we found that the minimal associate primes of $\mathcal{I}_7$ are the ideals $J_1, \ldots, J_7$ given above. So, we only have to show that any system from $\mathbf{V}(J_i)$ (for all $i = 1, \ldots, 7$) is linearizable.

For systems from $\mathbf{V}(J_1)$, $\mathbf{V}(J_2)$, $\mathbf{V}(J_3)$ the linearizing substitutions are presented in [4] (theorem 4.1) and systems from $\mathbf{V}(J_4)$, $\mathbf{V}(J_5)$ are subfamilies of the system VIII of [8, p 10 278] (the systems (2) and (3) of VIII, respectively). Therefore, it remains to consider the component $\mathbf{V}(J_7)$ ($\mathbf{V}(J_6)$ is mapped to $\mathbf{V}(I_7)$ by (1)), which means systems of the form

$$\dot{x} = x(1 - a_{10}x - a_{20}x^2) \qquad \dot{y} = -y(1 - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2). \tag{27}$$

When $a_{10}b_{01} \neq 0$, the transformation $x \to a_{10}x$, $y \to b_{01}y$ brings (27) to (16). The latter system has two invariant lines

$$l_1 = 1 - \frac{x}{2} - \frac{\sqrt{1 + 4a_{20}}x}{2} \qquad l_2 = 1 - \frac{x}{2} + \frac{\sqrt{1 + 4a_{20}}x}{2}$$

with the corresponding co-factors

$$K_1 = \frac{-x(1 + \sqrt{1 + 4a_{20}} + 2a_{20}x)}{2} \qquad K_2 = \frac{-x(1 - \sqrt{1 + 4a_{20}} + 2a_{20}x)}{2}.$$

Equation (13) has the solution

$$\alpha_1 = \frac{-1 - \sqrt{1 + 4a_{20}}}{2\sqrt{1 + 4a_{20}}} \qquad \alpha_2 = \frac{1 - \sqrt{1 + 4a_{20}}}{2\sqrt{1 + 4a_{20}}}.$$

Therefore the substitution

$$x_1 = x l_1^{\alpha_1} l_2^{\alpha_2}$$

linearizes the first equation.

According to lemma 1, system (3) has a first integral of the form (17). Due to proposition 2 of [5], this yields the existence of a first integral $\Psi(x, y)$ of the form (15). Therefore, the second equation is linearizable by the substitution

$$y_1 = \Psi(x, y) \big/ \big(x l_1^{\alpha_1} l_2^{\alpha_2}\big).$$

Obviously, the Zariski closure of $\mathbf{V}(J_7) \backslash \mathbf{V}(\langle a_{10}b_{01}\rangle)$ is equal to $\mathbf{V}(J_7)$. Therefore, any system from $\mathbf{V}(J_7)$ is linearizable. $\square$

Finally, we complete the classification of the one family of systems left outstanding from [8]. In theorem 3 of [8], there are two cases, (V)-(4) and (V)-(9), in which the authors could not show that the conditions were sufficient for linearizability of the system. These two cases are, in fact, dual under the transformation (1), so we only need to consider the first of these.

Taking $a = a_{-13}$ and $b = b_{10}$, the systems in class (V)-(4) can be written as

$$\dot{x} = x - 9bx^3 - ay^3 \qquad \dot{y} = -y + bxy - 6b^2x^2y. \tag{28}$$

(There is a misprint in the conditions (V)-(4) of [8]; it should be $a_{20}-9b_{10}^2$ instead of $a_{20}-9b_{01}^2$. The other case is stated correctly.)

We first perform the transformation

$$Y = y^3(1-3bx)^{-2} \qquad X = x - \frac{a}{4}Y \tag{29}$$

to bring the system (28) to the form

$$\dot{X} = X + 3bX^2 + \tfrac{9}{2}abXY + \tfrac{39}{16}a^2bY^2 \qquad \dot{Y} = -3Y - 6abY^2 \tag{30}$$

after scaling by the factor $(1-3bx)$. This system has a first integral in terms of hypergeometric functions from [3], proposition 4.15 (i). Using equation (29) to pull back the first integral of (30) to (28), we see that the original system must also be integrable. It is also clear from equation (30) that the line $1 + 2abY = 0$ is an invariant curve of the transformed system, whence $f = (1-3bx)^2 + 2aby^3 = 0$ gives an invariant curve of (28) with co-factor $-6bx - 18b^2x^2$.

Since equation (28) is integrable, there is a first integral of the form (15), $\Psi = xy + \cdots$. This means that equation (28) can be written as

$$\dot{x} = r\Psi_y \qquad \dot{y} = -r\Psi_x \tag{31}$$

for some analytic function $r = 1 + \cdots$ of x and y. Eliminating Ψ in equation (31) gives us

$$\dot{r} = r\,\mathrm{div}(\dot{x}, \dot{y}) = r(bx - 33b^2x^2). \tag{32}$$

We can therefore construct a function

$$\tilde{y} = yf^{1/8}r^{-1/4} \tag{33}$$

with $\dot{\tilde{y}} = -\tilde{y}$. As above, we can take $\tilde{x} = \Psi/\tilde{y}$, which satisfies $\dot{\tilde{x}} = \tilde{x}$, and the system (28) is linearizable.

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Appendix

To compute the linearizability quantities for systems (2) and (3) we can use the Mathematica code from [8, p 10291] with the two first lines replaced, respectively, by

```
11 [nu1_, nu2_, nu3_, nu4_, nu5_, nu6_, nu7_, nu8_] := 0 nu1 + 1 nu2 + 0 nu3
− 1 nu4 + 3 nu5 + 2 nu6 + 1 nu7 + 1 nu8;
12 [nu1_, nu2_, nu3_, nu4_, nu5_, nu6_, nu7_, nu8_] := 1 nu1 + 1 nu2 + 2 nu3
+ 3 nu4 − 1 nu5 + 0 nu6 + 1 nu7 + 0 nu8;
```

and

```
11 [nu1_, nu2_, nu3_, nu4_, nu5_, nu6_, nu7_, nu8_] := 1 nu1 + 2 nu2 + 0 nu3
− 1 nu4 + 3 nu5 + 2 nu6 + 0 nu7 + 0 nu8;
12 [nu1_, nu2_, nu3_, nu4_, nu5_, nu6_, nu7_, nu8_] := 0 nu1 + 0 nu2 + 2 nu3
+ 3 nu4 − 1 nu5 + 0 nu6 + 2 nu7 + 1 nu8;
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